Indian Statistical Institute, Bangalore

M. Math. Second Year

Second Semester - Partial Differential Equations

Mid-Semester Exam | Duration: 3 hours | Date : March 06, 2015

Answer all the questions. Max Marks: 25

1. Let $f \varepsilon L^1(R)$ be such that $\int f\varphi = 0$ for all φ in $C^{\infty}_{cpt}(R)$. Show that $f = 0$. [3]

2. Let X be a linear topological space where the topology is determined by a family $\{p_\alpha : \alpha \ \varepsilon \ I\}$ of semi norms. Let (Y, q) be a semi normed linear space. Let $T : X \to Y$ be linear map. If T is continuous, then show that there exists $\alpha_1, \alpha_2, ...\alpha_n$ and $c_1 > 0, c_2 > 0, ...c_n > 0$ such that [2]

$$q(Tx) \leq \sum_{1}^{n} c_j \ p_{\alpha_j}(x)$$

.

3. Let $P(\partial) = \sum_{j=0}^{k} a_j \ \partial^j, k \geq 2, a_k \neq 0$ and $a_0, a_1...a_k$ be complex numbers. Let $\eta_0 \ \varepsilon \ R$ be such that $P(i(\xi + i \ \eta_0))$ is never o for any ξ in R. Show that

   (a) $\int d\xi \frac{1}{|P(i(\xi+i \ \eta_0))|} < \infty$ [1]

   (b) Define $f(x) = \int d \ \xi \frac{e^{i \ x(\xi+i \ \eta_0)}}{P(i(\xi+i \ \eta_0))}$ [3]
   Show that f is a fundamental solution for $P(\partial)$.

4. Let Ω be open subset of R^n. $\psi_0, \psi_1 \varepsilon (\Omega|$. Such that $\psi_0 \ \psi_1 = \psi_1$ Let $f \ \varepsilon \ C^{\infty}_{cpt}(\Omega)$. Let $P(\partial)$ be any elliptic PDO. Let u be a tempered distribution on R^n such that $P(\partial)u = f$ on Ω. Assume that $\psi_0 u \ \varepsilon \ H^{so}(R^n)$. Show that $\psi_1 u \ \varepsilon \ H^{s_0+1}$ [5]

5. Prove Youngs Inequality viz $\|f * g\|_{L^2} \leq \|f\|_1 \|g\|_{L^2}$ [1]

6. (Prove Sobolev embedding lemma) $H^s(R^d) \ \subset \ C^{k_o}(R^d)$ if $s > \frac{d}{2} + k_0$
   Here $k_0 = 0, 1, 2, 3...$. [3]

7. For g in $\mathbb{S}(R^d)$ show that $gH^t \subset H^t$ [If you need Peetre inequality, $< \xi >^\sigma \leq 2^{|\sigma|/2}$ $< \eta >^\sigma < \xi - \eta >^{|\sigma|}$ you can use it]. [4]

8. Let $P(\partial)$ be elliptic PDO of order m. If $u \varepsilon H^s$ and $P(\partial)u \varepsilon H^s$ then show that $u \varepsilon H^{s+m}$. [3]